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A REVIEW OF THE METHOD OF DIMENSIONALITY REDUCTION IN CONTACT MECHANICS: APPLICATIONS FOR STRUCTURAL DAMPING, WEAR AND ADHESION

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Abstract: In the method of dimensionality reduction (MDR), contacts of three-dimensional bodies are mapped to the contact problem with a one-dimensional elastic or viscoelastic foundation. This is valid for the normal contact, the tangential contact and the normal contact of viscoelastic bodies. For the above classes of contact problems, several examples are considered and discussed in detail. This includes: (a) Fretting wear for arbitrary histories of loading (for simultaneous oscillations both in normal and horizontal directions); (b) Frictional damping under the influence of oscillations in normal and tangential direction as well as normal and torsional loading; (c) Adhesion of bodies of arbitrary axis-symmetric shape with extension to the adhesive contact of elastomers.

Keywords: Contact Mechanics, frictional damping, fretting wear, method of dimensionality reduction, adhesion

1. INTRODUCTION

Method of dimensionality reduction (MDR) is a method for solution of several classes of contact problems. It was developed at the Berlin University of Technology in a series of works since 2005. A detailed description can be found in the monograph [1]. In the framework of the MDR, the initial three-dimensional contact problem is replaced by a fictive contact problem of a specially defined one-dimensional series of springs (hence the notion "dimensionality reduction") with a plane profiles of a modified shape. The main statement of the MDR is that the results obtained with this "toy model" are exactly the same as for the initial true three-dimensional problem.

As often in the history of science, this new method is in reality based on the old results for the normal contact problem obtained by Galin (Russian Academy of Sciences) in the 1940th [2]. Sneddon translated and published the book of Galin and used later the theory of Galin for various applications. The theory of Galin became known to the western reader mostly through publications by Sneddon [3] and is often cited just as "Sneddon theory". The method of dimensionality reduction in its core version for axis-symmetric normal elastic contact is only a reformulation of the Galin-Sneddon solution to the form of a mnemonic rule which is easy to memorize and to handle. But it does not stick at the normal contact solution but generalizes this solution for many other contact problems. This generalization is based on a number of "equivalence principles" established in the course of development of the contact mechanics.

Thus, Cattaneo [4], and later Mindlin [5], showed that the tangential contact problem between parabolic profiles can be reduced to normal contact problem. Jäger [6] generalized this principle to arbitrary axis-symmetrical profiles and Ciavarella [7] proved several very general theorems – both for two-dimensional and three-dimensional contacts which all show that this reduction to the normal contact problem is a very general property of contacts of arbitrary shape. Adhesive contact problems of axis-symmetrical contacts can also be reduced to the normal contact problem without adhesion as shown by Galanov and Borodich [8-10]. This solution became more well-known through the paper by Yao and Gao [11], who directly used the Galin-Sneddon solution to derive the solution for adhesive contacts. The same is valid for contacts with viscoelastic contacts: As shown by Lee and Radok in the late 1950th, the contact viscoelastic bodies can be reduced to the normal contact of pure elastic bodies

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too [12-13]. The present paper is a review of the main procedures and some applications of the method of dimensionality reduction. In describing the method, we follow mainly the paper [14].

2. SOLUTION PROCEDURE IN THE FRAMEWORK OF THE MDR

Let us consider a contact between two elastic bodies with moduli of elasticity of E_1 and E_2, Poison's numbers of ν_1 and ν_2, and shear moduli of G_1 and G_2, accordingly. We denote the difference between the profiles of bodies as $z = f(r)$. In the framework of the MDR, *two* preliminary steps have to be conducted:

I. As already mentioned in the introduction, the elastic bodies of the initial three-dimensional contact problem are replaced by a one-dimensional linearly elastic foundation that is a linear series of independent springs (or spring-damper combinations), as shown in Figure 1. To describe the contact problem correctly, the normal stiffness Δk_z and the tangential stiffness Δk_x of individual springs have to be defined as follows:

$$\Delta k_z = E^* \Delta x \quad \text{with} \quad \frac{1}{E^*} = \frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2} \; , \; \Delta k_x = G^* \Delta x \quad \text{with} \quad \frac{1}{G^*} = \frac{2-\nu_1}{4G_1} + \frac{2-\nu_2}{4G_2} \; , \tag{1}$$

where Δx is the spacing between the springs, which is arbitrary but small enough to be able to resolve the essential structure of the contact.

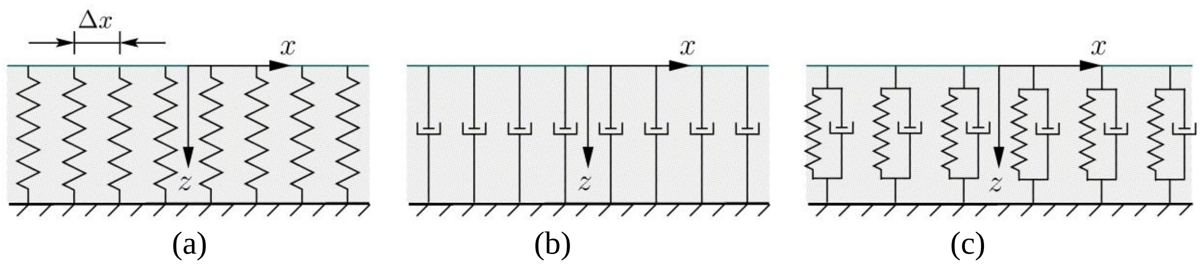

Figure 1. One-dimensional foundation of different materials: elastic foundation (a), purely viscous foundation (b), and viscoelastic foundation (c) with an example rheology according to Kelvin-Voigt.

Incompressible linearly viscous materials (pure fluids) can also be described by the MDR, but now the springs have to be replaced by dampers (Figure 1 b) with the damping coefficient $\Delta\gamma = 4\eta\Delta x$, where η is the dynamic viscosity of the medium. Similarly, arbitrary linear rheology can be represented as a series of independent "rheological elements" defined either as spring-damper combinations or in the general integral form. An example of such representation for the Kelvin body is shown in Figure 1c. Let us further note that for a small simplification of the following consideration, we will assume the condition of "elastically similarity" of material: $\frac{1-2\nu_1}{G_1} = \frac{1-2\nu_2}{G_2}$, to be fulfilled. This condition guarantees the independence of the normal and tangential contact problems [15]. It is important to note that it is always fulfilled for contact of bodies with the same elastic properties (as e.g. rail-wheel contact) or in a contact of a rigid body with an incompressible medium (as e.g. tire on the road).

II. The shape of the indenter also has to be changed. In the case of axis-symmetrical bodies this is done by the Galin-Sneddon-transformation

$$g(x) = |x| \int_0^{|x|} \frac{f'(r)}{\sqrt{x^2 - r^2}} \mathrm{d}r \; . \tag{2}$$

The reverse transformation is

$$f(r) = \frac{2}{\pi} \int_0^r \frac{g(x)}{\sqrt{r^2 - x^2}} \mathrm{d}x \, . \tag{3}$$

Through this step, the initial three-dimensional profile $z = f(r)$ (Figure 2, left) is replaced by a plane profile of another, strictly determined, form (Figure 2, right)

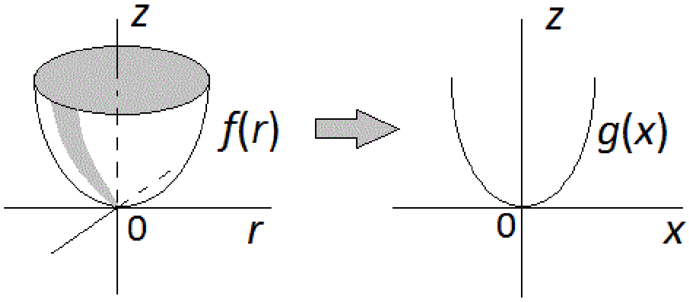

Figure 2. The three-dimensional profile has to be replaced by a plane profile using the MDR-transformation.

3. FRETTING WEAR

At the first glance, in the MDR-model, information about the detailed distribution of stresses and deformations in and outside of the contact gets lost. In reality, this is not the case. The distribution of normal pressure p in the initial three-dimensional problem can be restored using the following integral transformation [1]:

$$p(r) = -\frac{1}{\pi}\int_r^\infty \frac{q'(x)}{\sqrt{x^2 - r^2}}\mathrm{d}x = \frac{E^*}{\pi}\int_r^\infty \frac{g'(x)}{\sqrt{x^2 - r^2}}\mathrm{d}x \qquad (4)$$

If the profile is periodically moved tangentially with the amplitude $u_x^{(0)}$, the bodies will be worn in a circular area (fretting) and the shape will tend to some limiting shape [16]. The shape of the limiting form can be found very elegantly using the MDR. First, it is clear that it will be some sticking region where there is no relative motion of two bodies. The condition for the radius of this stick region has been found already by Cattaneo and Mindlin. In the framework of MDR this can be done in the following way: If we move the indenter tangentially, the springs become stressed (both in the normal and tangential direction). The radius c of the stick region is determined by the condition that the tangential force $k_x u_x^{(0)}$ is equal to the coefficient of friction μ multiplied with the normal force $k_z u_z(c)$, thus, $G^* u_x^{(0)} = \mu E^*\left(d - g(c)\right)$.

As there is no relative movement in the stick region, there will be no wear, and the profile in this region just coincides with the initial profile. Now we go over to determining the worn shape outside the stick region. Let us denote the initial three-dimensional profile as $f_0(r)$. The corresponding one-dimensional image be $g_0(x)$. The limiting "shakedown shapes" are denoted as $f_\infty(r)$ and $g_\infty(x)$. The wear in the slip region will proceed as long as the surfaces are pressed against each other in this region. Thus, the limiting shape is defined by the condition that pressure disappears [16]: $p(r) = 0$, for $r > c$. From (4), it follows that

$$g'(x) = 0 \ \text{ and } g(x) = const = g_0 \ \text{ for } \ c < x < a \qquad (5)$$

Outside the worn region, the body still will have the initial shape, and the boundary of the worn and non-worn regions will be indented exactly by the macroscopic indentation depth. It then follows that the one-dimensional profile in the limiting state has the form

$$g_\infty(x) = \begin{cases} g_0(x), & \text{for } 0 < x < c \\ d, & \text{for } c < x < a \end{cases} \qquad (6)$$

The three-dimensional limiting shape has nor to be calculated by applying the transformation (3). An example of the worn form for some particular values of parameters is shown in Figure 3.

$$f_\infty(r) = \begin{cases} f_0(r), & \text{for } 0 < x < c \\ \dfrac{2}{\pi}\displaystyle\int_0^c \frac{g_0(x)}{\sqrt{r^2 - x^2}}\,\mathrm{d}x + \frac{2}{\pi} d \int_c^r \frac{1}{\sqrt{r^2 - x^2}}\,\mathrm{d}x, & \text{for } c < x < a \end{cases} \tag{7}$$

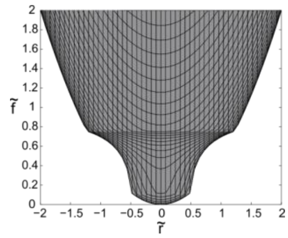

Figure 3. Worn shape of the final "shakedown" state.

4. APPLICATION OF THE MDR TO ADHESIVE CONTACT

The calculation procedure in the case of adhesive contacts is very similar to that for non-adhesive normal contacts. First, the same two steps of transforming the materials into one-dimensional elastic foundation and shape transformation have to be performed. Now the MDR-transformed planar shape is pressed into the elastic foundation and pulled then back. We first assume that all of the springs in the contact adhere to the indenter, so that the contact radius remains the same. However, if the elongation of the boundary springs achieves some definite critical value, they detach. The rule for the detaching was found by M. Heß and is known as the rule of Heß:

$$\Delta\ell(\pm a) = \Delta\ell_{\max}(a) := \sqrt{\frac{2\pi a\,\Delta\gamma}{E^*}}\,. \tag{8}$$

Here, $\Delta\gamma$ is the separation energy of the contacting bodies per unit area. It can be shown that this state corresponds exactly to the equilibrium state of the three-dimensional adhesive contact [19]. That means, that all relations between the normal force, the indentation depth and the contact radius, which will be obtained in the MDR model will represent exact solutions for the initial three-dimensional adhesive contact problem. The radius of the contact determined by the condition that the displacement of springs at the boundary of the contact is equal to the critical value:

$$u_z(\pm a) = -\Delta\ell_{\max}(a) \quad \Rightarrow \quad d := g(a) - \Delta\ell_{\max}(a). \tag{9}$$

The normal force is given as before by Equation

$$F_N = 2E^*\left[\int_0^a x g'(x)\,\mathrm{d}x - a\Delta\ell_{\max}(a)\right]. \tag{10}$$

Two classical cases considered in contact mechanics are the cases of controlled force and controlled displacement. In the case f controlled force, the critical value a_c of the contact radius at the moment of the loss of stability is determined by the condition $\mathrm{d}F_N / \mathrm{d}a = 0$: $\dfrac{\mathrm{d}g(a)}{\mathrm{d}a} = \sqrt{\dfrac{9\pi\Delta\gamma}{2aE^*}}$. Inserting the critical radius obtained from this equation into (10) results in the maximum negative force. We will call its magnitude the adhesion force F_A:

$$F_A = 2E^*\left[a_c\Delta\ell_{\max}(a_c) - \int_0^{a_c} x g'(x)\,\mathrm{d}x\right] \tag{11}$$

Just to illustrate the extreme simplicity of the MDR calculations, let us consider adhesion of a plat cylindrical indenter of radius a with an elastic half space. In this case, the integral in (11) is equal to zero and the adhesion force is only given by the first term: $F_A = 2E^* a\Delta\ell_{\max}(a) = \sqrt{8\pi a^3 E^* \Delta\gamma}$, which corresponds to the result of Kendall [17].

5. FRICTIONAL DAMPING

In [17-18], it was shown that if a contact oscillates in both normal and tangential directions, there is a purely elastic loss mode that was called "relaxation damping". In the present section we follow the publication [18]. Let us consider a contact of two elastic bodies. As is well known, this problem can be reduced to a contact of a rigid indenter and a correspondingly renormalized elastic half-space. Let the profile oscillate harmonically with a normal amplitude $u_z^{(0)}$, a tangential amplitude $u_x^{(0)}$ and a phase difference φ_0. To illustrate the essence of the relaxation damping, we consider the case of very high coefficient of friction, so that the bodies are assumed to stick in all points in which they are in contact. Under this condition, there will be no relative motion of bodies at any time, but we will show that there will still be a finite energy dissipation. While in the three-dimensional space, this is a very complicated problem, it becomes very simple in the MDR-representation. Indeed, as the elements of the MDR model are independent springs, it is enough to consider only one spring, as shown in Figure 4, and then to summarize over all springs which come into contact during an oscillation cycle. Note that there are three categories of springs in the system: those which are always in contact – they do not contribute to energy dissipation, those which are never in contact – they of course also do not contribute to energy dissipation, and finally those in "intermittent contact" – which come into contact, are dragged by the indenter and then loose the contact again. Only these springs contribute to energy dissipation. Consider a point of the rigid indenter with initial distance $z^{(0)}$ from the "surface". Its coordinates during the oscillatory motion can be written as $z(t) = -z^{(0)} + u_z^{(0)} \cos \omega t$ and $x(t) = x^{(0)} + u_x^{(0)} \cos(\omega t + \varphi_0)$. If $\left|u_z^{(0)}\right| > z^{(0)}$, the point of the rigid surface will come into contact with one of the springs of the elastic foundation in point x_1 and will drag it along to point x_2, where contact is lost and the spring relaxes over the distance $s = x_2 - x_1$. The coordinates x_1 and x_2 are determined by setting $z = 0$ which provides $s = x_2 - x_1 = 2u_x^{(0)} \sqrt{1 - \left(z^{(0)} / u_z^{(0)}\right)^2} \sin \varphi_0$.

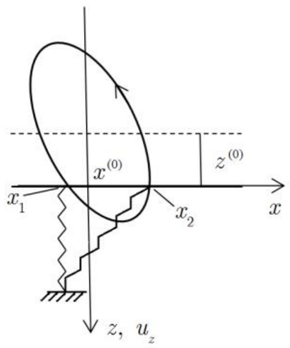

Figure 4. A point of the rigid surface with the initial coordinate $z = -z^{(0)}$ oscillates around this position. It comes into contact with a spring in point x_1 and loses contact in point x_2.

The energy dissipated by a single spring during one cycle is equal to the energy stored in the stressed spring at the time of its release: $\Delta W = \frac{1}{2}\Delta k_x s^2 = \frac{1}{2} G^* s^2 \Delta x$. As stated above, energy dissipation occurs only if the spring of the foundation was in contact with the substrate during only a part of the cycle. This is the case for all points which satisfy the condition $-\left|u_z^{(0)}\right| < z^{(0)} < \left|u_z^{(0)}\right|$. Substituting $\Delta x = \Delta z^{(0)} / c$ in equation for the dissipated energy (where c is the slope of the equivalent profile at the edge of the contact) and integrating over the above stated interval, we obtain the total dissipated energy per cycle:

$$W = 2\frac{1}{2}\frac{G^*}{c}\int_{-\left|u_z^{(0)}\right|}^{\left|u_z^{(0)}\right|} s^2 dz^{(0)} . \tag{12}$$

The factor "2" takes into account that there are two symmetric regions on both sides of the contact giving equal contributions to dissipation. Evaluation of the integral finally gives

$$W = \frac{16}{3}\frac{G^*}{c}u_x^{(0)2}\left|u_z^{(0)}\right|\sin^2\varphi_0 \tag{13}$$

or in the shape invariant form (details see in [18])

$$W = \frac{8}{3}\frac{G^*}{E^*}\frac{\partial^2 F_N}{\partial d^2}u_x^{(0)2}\left|u_z^{(0)}\right|\sin^2\varphi_0 . \tag{14}$$

6. CONCLUSION

The present paper is an illustration of the simplicity of the method of dimensionality reduction in practical handling. Its main rules can be explained (without formal proof) in a couple of lines and the application to particular problems is as simple as the general structure of the method. Those who would like to see the proofs of the correctness of the MDR are referred to the original publications [1] and [19].

Here we confined ourselves to consideration of axis-symmetrical contact problems. This is the class of problems where the method can be applied in the easiest manner. However, the method is not restricted to axis-symmetrical contacts. Thus in [20, 21], it was shown that it can be successfully applied to rolling, which is not completely axis-symmetric problem. It can be applied to contacts of rough surfaces [23, 24, 29] if only the macroscopic response is of interest (force-displacement relations). It can be applied to contacts with elastomers [22]: We stated this already in the Introduction but did not have space in the frame of this publication to illustrate these applications. The MDR can be applied in an approximate manner to elastomer friction [25, 26], to thermal effects in contacts [1], acoustic emission in rough contacts [27]. One of the main advantages of the MDR is the extreme numerical effectiveness which allows to incorporate the contact simulation into system dynamic simulations [28].

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